A Market Framework for Pooling Opinions

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Abstract

For decades, scientists have grappled with how best to aggregate subjective probabilities from multiple experts in order to form consensus assessments. Many combination functions or *opinion pools* have been proposed, each sensible under particular assumptions or measures. We examine a market-based approach, where the equilibrium prices of *securities*, paying off contingent on uncertain events, are interpreted as consensus probabilities. We develop a prescriptive model, identifying correspondences between the consensus probabilities derived by the market and by the traditional opinion pools: parameters of the agents' utilities in the former context serve the role of expert weights in the latter. In an extended model, in which agents learn from prices, the weights are shown to additionally depend on self-assessed confidence. Even when agents themselves do not learn, we show that the market as a whole can adapt over time in the manner of a Bayesian-rational individual, by transferring wealth from inaccurate to accurate agents.

1 Introduction

The notion that a rational decision maker should behave according to a subjective probability distribution is widely accepted. Defining an anal-

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ogous concept for a *group* of decision makers has met with more controversy. Many *belief aggregation* methods, or opinion pools, have been proposed to derive consensus probabilities for groups. Forming consensus beliefs is a classic problem in statistics [11, 17, 23, 36, 80, 81, 82], decision analysis [4, 5, 6, 12, 46, 47, 49, 84, 85, 86], and has found application in many areas, from group decision making [45, 78, 90] to artificial intelligence [30, 39, 41, 42, 53, 56, 57, 88, 89]. In some group situations there is a natural leader ultimately responsible for decisions, and standard techniques may apply [37]. Other groups are more egalitarian; in these situations, understanding what beliefs can be the object of a consensus, and how collective decisions can be made, may be of considerable interest. Despite decades of research on opinion pools, including a history of contributions reported in this journal [2, 35, 39, 48, 52, 71, 73], definitive answers have been elusive, and various impossibility results [9, 19, 22, 29, 34, 77] and definitional controversies [17] in the literature cast doubt on the prospects for an entirely satisfactory solution. Much debate revolves around which properties of the aggregate distribution are "reasonable", and which pooling functions exhibit them. We take instead a bottom-up approach, beginning with characterizations of individual behavior, and deriving the implications for aggregate properties from there.

One of the most convincing justifications for *individual* subjective probabilities is de Finetti's famous *no Dutch book* argument [10]. Imagine that a bettor B must decide at what price to buy and sell each of two lottery tickets: one pays \$1 if an event A_1 occurs, and nothing otherwise; the other pays \$1 if A_2. If A_1 and A_2 are disjoint events, then B's total price for both tickets should never exceed \$1—otherwise, B is exposed to a guaranteed loss: a shrewd opponent O will sell both tickets to B for more than \$1, paying out at most \$1 in all outcomes. More generally, if B's internal prices for a set of such lottery tickets are not legal probabilities, then B is subject to certain losses—also called *Dutch books* or *arbitrage* [75]. Now imagine that a *group* of bettors, $B_1, B_2, \ldots$ are locked in a room. O is outside, and slides offers to buy and sell lottery tickets under the door. O does not care who accepts the offers: if opportunities for risk-free profit exist, O will take advantage of them. From O's perspective, such opportunities imply that the group or "room" is irrational. To be concrete, imagine that B_1 will sell "\$1 if A_1" at \$0.5, and B_2 will buy the same ticket at \$0.7. O could potentially buy a ticket from B_1 and sell it to B_2 for a riskless profit of \$0.2. But if the group had first exhausted all opportunities for within-group trades, then B_1

would sell instead to B_2. Each bettor's transactions are exactly the same, and the group as a whole is better off by \$0.2. Only if there is *excess demand* within the group for some lottery ticket at some price (unmatched offers to buy or sell), should they accept a countering outside offer. If the group is in an *equilibrium* state—that is, all possible within-group trades have been exhausted—then arbitrage opportunities cannot exist [51]. From O's perspective, the group appears rational. In other words, by de Finetti's same argument, the group behaves as if guided by a single *representative* probability distribution.

The hypothetical group of bettors in the above example essentially created a market in lottery tickets. Each bettor acts rationally, buying and selling according to his own beliefs and utility for money. The equilibrium prices in the market can be interpreted as the group's consensus probability distribution. In this paper, we examine the reasonableness of this interpretation under various market modeling assumptions.

In Section 2, we cover necessary background material on opinion pools and markets, and introduce our basic market framework. In Section 3, we show that under each of two homogeneity assumptions regarding agents' utility types, equilibrium prices have the same functional form as standard opinion pools. The so-called *expert weights*, notoriously hard to define in the context of opinion pools [17], correspond to parameters of the agents' utility functions. In Section 4, we present an extended model where agents update their beliefs based on market signals, by treating prices as if they were the beliefs of another individual. In this case, expert weights depend not only on utilities, but on assessments of confidence as well. In Section 5, we describe a situation where the market itself adapts over time; the redistribution of wealth among agents leads to price dynamics that actually mimic rational Bayesian updates. In Section 6, we describe previous proposals for market-based belief aggregation and discuss related work in experimental economics. Some final remarks appear in Section 7.

2 Background, Notation, and Framework

We presume that N agents are uncertain about which of many possible world *states* will eventually obtain. Denote the set of possibilities as $\Omega = \{\omega_1, \omega_2, \ldots\}$, where the ω are disjoint and exhaustive. There are S *events* of interest (i.e., events for which a consensus is to be reached), $A_1, A_2, \ldots, A_S$,

where each A_j is a subset of Ω. Agent i has a subjective probability distribution $\Pr_i(\cdot)$ over Ω; its probability for an event A_j is then just $\Pr_i(A_j) = \sum_{\omega \in A_j} \Pr_i(\omega)$. In general events can overlap, though we consider in detail the special case of *mutually exclusive* events. Agent i's utility for y dollars is $u_i(y)$. When only one agent or one event is under consideration, we simplify by dropping the appropriate subscript(s).

An *opinion pool* combines the probabilistic beliefs of a group into a single, consensus belief, denoted $\Pr_0$.

$$\Pr_0 \equiv f(\Pr_1, \Pr_2, \ldots, \Pr_n)$$

2.1 Opinion Pools

A variety of authors have proposed or advocated a corresponding variety of aggregation functions—Genest and Zidek [23] and French [17] provide comprehensive surveys. Three approaches are generally distinguished. The first assumes a single Bayesian decision maker h (real or fictitious, within or outside the group), called the *supra Bayesian*, with a joint distribution over all events and all participants' beliefs. The supra Bayesian updates its beliefs via Bayes's rule, given the "evidence" of everyone else's beliefs. The resulting posterior is taken to be the consensus belief,

$$\Pr_0(\omega | \Pr_1, \ldots, \Pr_N) \propto \Pr_h(\Pr_1, \ldots, \Pr_N | \omega) \Pr_h(\omega). \tag{1}$$

The second approach is to apply a prespecified function, usually some form of weighted average, that maps any set of probability distributions to a single distribution. Note that some such functions can be interpreted as the the updating procedure of a supra Bayesian. Each pooling function is usually justified axiomatically, by assuming a "reasonable" set of properties of the aggregate distribution. The two most common and well-studied aggregation functions are the linear and logarithmic opinion pools (LinOP, LogOP). The LinOP is a weighted arithmetic mean of the members' probabilities,

$$\Pr_0(\omega) = \sum_{i=1}^{N} \alpha_i \Pr_i(\omega), \tag{2}$$

and the LogOP is a normalized, weighted geometric mean,

$$\Pr_0(\omega) \propto \prod_{i=1}^{N} [\Pr_i(\omega)]^{\alpha_i}, \tag{3}$$

where the α_i are called *expert weights*, usually nonnegative numbers that sum to one. The LinOP and LogOP can actually be characterized as two instances of a parameterized family of weighted aggregation functions [7]. A third, relatively new, approach to pooling opinions is based on maximum entropy inference. The consensus is the unique probability distribution that maximizes Shannon entropy, chosen from among the distributions that are consistent with all available information, including the experts' beliefs, their past performance, and/or dependencies among experts [35, 49].

The debate over which aggregation method is best continues to rage [2, 7, 30, 52, 86]. Several authors (most emphatically Lindley [36, 37]) argue that the supra Bayesian approach is superior, as it is grounded in standard normative Bayesian theory [6, 45, 46, 82, 85, 66]. However, several conceptual questions arise in characterizing the *single* supra Bayesian responsible for the consensus of a (possibly leaderless) group [23]. If the supra Bayesian is real, is it a member of the group or an outsider? If an outsider, will the group accept its authority to generate the consensus? If a member, which one? If it is fictitious, where do its prior probabilities come from? These must somehow be agreed upon in advance, but then how is *that* consensus reached? A second problem is analytic tractability—the supra Bayesian must have a joint distribution over all members' beliefs and the events in question. In almost every treatment of this method, strong assumptions are made for the supra Bayesian's priors and likelihood function—for example, by assuming that the members' log-odds are normally distributed and independent—in order to yield tractable results [6, 30, 37, 85, 66]. Weighted aggregation functions are not without their own drawbacks. Each method implies different properties for the combined distribution. Many of these properties seem reasonable, but disagreement persists on which are essential.[4] Each of many opinion pools, including the LinOP and LogOP, can be shown to be the *only* form possible, if one assumes an appropriate, corresponding set of axioms [4, 47, 20, 18, 21, 80]. It follows that no pool can satisfy a sufficiently large set of potentially desirable properties, resulting in several so-called *impossibility* theorems [8, 9, 19, 22, 34, 77]. Another point of contention is how best to determine the expert weights. In most cases they are chosen in an ad hoc manner to encode some measure of confidence, reliability, or importance [2, 17, 84].

[4]For example, Lindley [36] regards the so-called *marginalization property* as an "ad-hockery" while Cooke [7] characterizes any consensus function that does *not* respect it as "downright queer".

Some more formal methods to derive weights have been proposed, by making assumptions concerning the form of, or interdependence among, participants' beliefs [7, 12, 30, 46], or through iterative self-weighting procedures [11].

In all three categories of opinion pools, most work presumes—sometimes unjustifiably [17]—that agents are honest in reporting self-assessed weights and probabilities, or that standard techniques for eliciting truthful assessments from individuals [10, 70, 87] can extend to the group setting.

2.2 Securities Markets and Information Aggregation

In certain real markets, a common interpretation of prices is as an amalgamation of opinions. For example, the price of a stock represents the "market evaluation" of the expected present value of future dividends, and odds in a horse race summarize the bettors' beliefs about the winning horse's identity [72, 79]. Under specific regularity conditions, the exact dependence of prices on beliefs can be characterized [28, 67, 69, 83], and compared with the opinion pools of Section 2.1. In contrast to the Bayesian or axiomatic approaches, economic models are usually built upon behavioral assumptions (e.g., that agents are rational utility maximizers) and descriptions of the price-formation process.

In a classical, pure-exchange economy *without* uncertainty, agents have *endowments* of commodities or *goods* and exchange them according to their utility for consumption. Under uncertainty, endowments and utilities are in general state-dependent, and thus so are the spot prices of goods. Agents will generally exhibit varying marginal value of wealth in the different states. Consequently, if they are risk-averse, they will desire to *hedge* or *insure* against risk by distributing wealth across states.[5] The *Arrow-Debreu securities market* is the fundamental theoretical framework in economics and finance for resource allocation under uncertainty [1, 13, 40]. A *security*, denominated in money or other exchangeable good, pays off variously contingent upon the realization of an uncertain state. In the standard model, agents trade securities prior to revelation of the world state, and then trade real goods once the state is known. A market is termed *complete* if it contains at least $|\Omega| - 1$ *linearly independent* securities,[6] plus spot markets in

[5] For example, insuring a package delivery transfers wealth from the *package-delivered* state to the *package-lost* state.

[6] This is often realized with one security paying out in each state of nature, though any set whose payoff-by-state matrix has rank $|\Omega| - 1$ will do.

the original goods. Such a market guarantees, under classical assumptions, that equilibrium entails a Pareto optimal allocation of both goods and risk.

When one unit of each security pays out one dollar, it follows, from de Finetti's argument, that the equilibrium prices in a securities market form a legal probability distribution [13]. Moreover, all agents' *risk-neutral* probabilities are in agreement, and coincide with these prices [13, 31, 51]. Risk-neutral probabilities are defined as $\Pr^{\mathrm{RN}}(\omega) \propto \Pr(\omega)u'(Y_\omega)$, where Y_ω is the agent's payoff in state ω and u' is the derivative of utility [50]. A hypothetical agent with transformed probabilities $\Pr^{\mathrm{RN}}(\omega)$ and transformed (state-dependent) *risk-neutral* utility $u^{\mathrm{RN}}(y) \equiv u(y)/u'(Y_\omega)$ has precisely the same betting behavior as an agent with $\Pr(\omega)$ and $u(y)$. Since any two risk-neutral agents with differing probabilities will desire to trade, the market cannot be in equilibrium unless all agents' risk-neutral probabilities agree, and equal the prices [51].

To derive explicit equations for price, early work centered on the *prior information* (PI) equilibrium concept, where agents' beliefs are presumed to remain fixed during price formation [28, 67, 69]. As discussed in Section 3, prices can in this case be characterized a combination of the agents' beliefs, but cannot reasonably be regarded as an accumulation of diverse *information*. The latter, perhaps more satisfying, interpretation does become admissible under the *rational expectations* (RE) equilibrium assumption [24, 38]. This theory asserts that prices actually reflect all of the information available to all agents. Even when some agents have exclusive access to inside information, prices equilibrate exactly as if everyone had access to all information. The procedural explanation is that prices *reveal* to the ignorant agents any initially private information; that is, agents *learn* from prices. Several authors show that, if agents begin with identical priors and disparate evidence, repeated observation of some aggregate statistic (e.g., price) will converge to a consensus on posteriors, for various sufficient statistics [25, 43, 44, 54]. We describe a supra Bayesian model of RE in Section 4.

Both Rubinstein [67] and Wilson [83] develop formal conditions, examined further in Section 3.1.3, under which agents in a securities market behave collectively in a manner indistinguishable from a single, *composite* agent whose beliefs equal the equilibrium prices [28]. Raiffa [64] (pp. 211–216) recounts an example, due to Pratt, where the optimal group action requires randomization, and thus a composite agent *cannot* exist. Nevertheless, Kasanen and Trigeorgis [32] demonstrate empirically that the analogy of a market as a representative investor can sometimes apply, even outside the theoreti-

cal bounds. The composite-agent construction constitutes perhaps the most compelling interpretation for prices as consensus beliefs.

2.3 A Securities Market Model for Belief Aggregation

Several economic constructs central to the study of securities markets (e.g., spot markets in real goods, consumption, production, and endowments) are extraneous in a model focusing on belief aggregation. In what follows, we develop a stripped-down model sufficient for our purposes.

The market consists of securities "\$1 if A_j", abbreviated $\langle A_j \rangle$, for every event of interest A_j. One unit of the security pays off one dollar if the event occurs, and nothing otherwise. If an agent must pay a price p per unit of security $\langle A \rangle$, then purchasing x units is equivalent to accepting a lottery with payoff $(1-p)x$ if A occurs, and $-px$ otherwise. Positive x indicates a quantity to buy, and negative x a quantity to sell. *Either* action incurs risk: buying $\langle A \rangle$ at price p yields exactly the same lottery as selling $\langle \bar{A} \rangle$ at price $1-p$.

Agents are *competitive*; that is, they maximize expected utility at a given set of prices, or equivalently, ignore any effect their own demand may have on prices. We assume that utility for money is independent of state and monotonically increasing. To avoid unbounded demand, we further assume that agents are *risk averse* [33, 62], that is, have concave u, always preferring the expected payoff of a lottery to the lottery itself. Before trading begins, the agents do not have any prior stakes in the events (e.g., they do not own any securities from outside of the market). We restrict attention to linearly independent securities (i.e., none can be replicated as a combination of others).[7] Without this, agents could buy one security and sell the equivalent portfolio without risk, and demand may be nonunique or unbounded.[8] We consider the PI and RE equilibrium conditions in Sections 3 and 4, respectively.

In a market of S securities, let $x^{(j)}$ represent a quantity of security $\langle A_j \rangle$, and $p^{(j)}$ the security's price. The agent's utility for a bundle $\mathbf{x} = \langle x^{(1)}, x^{(2)}, \ldots, x^{(S)} \rangle$ of securities is its expected u:

$$U(\mathbf{x}) = \sum_{\omega \in \Omega} \Pr(\omega) u(Y_\omega), \tag{4}$$

[7] This assumption implies that each event is strictly uncertain ($0 < \Pr(A) < 1$).

[8] Note that the equilibrium price of any linearly dependent security is computable using no-arbitrage arguments.

where

$$Y_\omega = \sum_{j=1}^{S} (1_{\omega \in A_j} - p^{(j)}) x^{(j)}$$

is the payoff in state ω, with $1_{\omega \in A_j}$ the indicator function that equals one if $\omega \in A_j$, zero otherwise.

The agent's *demand function*, $\mathbf{x}(\mathbf{p}) = \langle x^{(1)}(\mathbf{p}), x^{(2)}(\mathbf{p}), \ldots, x^{(S)}(\mathbf{p}) \rangle$, represents the quantities of securities that maximize utility (4) at prices $\mathbf{p} = \langle p^{(1)}, p^{(2)}, \ldots, p^{(S)} \rangle$. Concavity of u (risk aversion) entails concavity of U, and hence any critical point of (4) is a global maximum. An agent whose beliefs differ from the going prices will desire to trade, and its choices will always yield, from its perspective, nonnegative expected payoff. Agents are given unlimited borrowing and short-selling privileges, though the quantities $\mathbf{x}(\mathbf{p})$ demanded will be naturally bounded as long as utility is sufficiently convex[9] and securities are linearly independent.

Equation 4 captures the optimization problem of a single agent faced with the decision to invest in securities. In an economy of N agents, each continually maximizing (4), prices adjust until all buy orders (positive $x^{(j)}$) match with sell orders (negative $x^{(j)}$) for each security $\langle A_j \rangle$. A market is in *competitive equilibrium*[10] at prices $\mathbf{p}$ if and only if

$$\sum_{i=1}^{N} \mathbf{x}_i(\mathbf{p}) = \mathbf{0}. \tag{5}$$

If all agents maintain identical beliefs across all events, then equilibrium prices will equal these beliefs. This property, called *unanimity*, is often considered a minimal standard of acceptability for an opinion pool.

Most opinion pools require (or assume) that agents directly, and truthfully, report their probabilities to some centralized coordinator. Scoring

[9] Risk aversion alone does not guarantee bounded demand. One sufficient condition is that $\lim_{y \to \infty} \frac{u(y)}{u(-y)} = 0$.

[10] This formulation assumes a Walrasian environment, where exchange occurs only after equilibrium is reached. In an alternative formulation, agents make incremental trades for securities, at mutually beneficial prices, until no such opportunities exist. Under our assumptions, such a process would also reach an equilibrium, where agents' risk-neutral probabilities are equal (and thus represent clearing prices). Solutions for equilibrium prices for the case of negative exponential utility (derived below) would remain the same, though our other analytic results are specific to the Walrasian formulation. The dynamic model of Section 5 reflects some aspects of the incremental approach.

rules [70, 87], promissory notes [10], or other incentive devices could be leveraged to elicit honest assessments, though these may require outside subsidies and, as French [17] points out, a group may collectively have greater incentive to lie. Kadane and Winkler [31] demonstrate that the probabilities revealed by such elicitation aids are inextricably confounded with utility, whenever the agent is risk averse and may have stakes in the events. The securities market framework provides built-in rewards for participation, honesty,[11] and the collection of cost-effective evidence. Agent beliefs are revealed only implicitly through their trades. In fact, if measures are taken to ensure the privacy of bids, the *only* personal information directly revealed is an agent's risk-neutral probability distribution at equilibrium, which equals the consensus anyway. Since supply and demand are balanced in equilibrium, the mechanism requires no subsidy, and so the only cost involved is that of organizing and running the market.

3 Market Pooling Functions

In this section, we characterize the dependence of prices on beliefs within our model, concentrating first on the case where all events are mutually exclusive. We consider, in turn, two specific assumptions regarding agents' utility forms.

3.1 Disjoint Securities

Events are disjoint but, to retain linear independence, are not exhaustive. For notational purposes, define the event $A_0 \equiv \bar{A}_1 \bar{A}_2 \cdots \bar{A}_S$, though it does not have a corresponding security.[12] Mutually exclusive events are certainly a special case, however any event space can partitioned into securities representing disjoint events.

3.1.1 Negative Exponential Utility

First, suppose that all agents have negative exponential utility for money, or $u(y) = -e^{-cy}$. This utility is synonymous with *constant absolute risk*

[11] Incentives exist, though the market is not fully *incentive compatible* [40].

[12] $\langle A_0 \rangle$ can be replicated by selling equal amounts of all securities.

aversion (CARA), where c is the coefficient of risk aversion, or $1/c$ the risk tolerance.

Each agent's unique optimal demand is:

$$x^{(j)}(\mathbf{p}) = \frac{1}{c} \ln \left(\frac{\Pr(A_j)}{p^{(j)}} \cdot \frac{1 - p^{(1)} - p^{(2)} - \cdots - p^{(S)}}{\Pr(A_0)} \right). \tag{6}$$

The condition for competitive equilibrium (5) is $\sum_{i=1}^{N} x_i^{(j)}(\mathbf{p}) = 0$, for all $j = 1 \ldots S$. Solving for $\mathbf{p}$, we arrive at the following proposition.

Proposition 1 (CARA $\Rightarrow$ LogOP) *In a market of disjoint securities, each agent having CARA c_i, the prices*

$$p^{(j)} = \frac{\prod_{i=1}^{N} [\Pr_i(A_j)]^{\alpha_i}}{\prod_{i=1}^{N} [\Pr_i(A_0)]^{\alpha_i} + \prod_{i=1}^{N} [\Pr_i(A_1)]^{\alpha_i} + \cdots + \prod_{i=1}^{N} [\Pr_i(A_S)]^{\alpha_i}} \tag{7}$$

form a competitive equilibrium, where $\alpha_i = (1/c_i)/\sum_l (1/c_l)$ is the normalized risk tolerance of agent i.

Proof. These prices can be shown to satisfy (5) by substitution, and some lengthy algebra. □

These prices are the same probabilities as derived by the LogOP (3), where the expert weights are normalized risk tolerances. The market model, therefore, provides one way to ground a well-known centralized pooling mechanism in terms of individual behavior. Proposition 1 can be thought of as a corollary to Rubinstein's result [67] within our context, the major difference being that the normalization of prices falls out naturally in our derivation, yielding a legal probability distribution. Wilson [83] also arrives at the weighted geometric average relationship when agents (with CARA) share the profits of a security that pays off contingent on a single, continuous random variable. Nau and McCardle [51] derive a similar result for more than one continuous variable when all beliefs are multivariate normal, and quadratic options are available in addition to the standard securities.

Proposition 1 provides a decision-theoretic interpretation for the notoriously slippery concept of expert weights. In the usual interpretation, the exponents in (7) encode some sort of degree of expertise, confidence, or reliability, and are almost always chosen in an ad hoc manner [2, 17, 23, 53, 84]. In the market model, weights emerge from agent characteristics, and are directly related to a well-understood quantity, namely risk aversion. It may

seem that weights *should* relate to a notion of confidence, however loosely defined, rather than to risk tolerance. In Section 4, we propose an extension of the market model in which agents learn from prices, and show how a confidence-related interpretation arises naturally.

3.1.2 Generalized Logarithmic Utility

Next consider an assumption that all agents exhibit *generalized logarithmic utility* (GLU) for money: $u(y) = \ln(y + b)$. With the further constraint that $b > 0$, GLU is a standard *decreasingly* risk-averse form. Note that u is imaginary on the interval $(-\infty, -b)$. Since utility approaches $-\infty$ as $y \to -b$ from the right, and u is increasing, we can safely extrapolate that it is $-\infty$ for all $y \leq -b$. The upshot is that the agent would *never* put b or more dollars at risk, regardless of probability assessments. Note that any wealth accumulated in dollars is additive with b, so a natural interpretation of b is as the agent's initial wealth.

Unique optimal demand is

$$x^{(j)}(\mathbf{p}) = b\left(\frac{\Pr(A_j)}{p^{(j)}} - \frac{\Pr(A_0)}{1 - p^{(1)} - p^{(2)} - \cdots - p^{(S)}}\right), \tag{8}$$

and equilibrium prices can again be characterized in closed form.

Proposition 2 (GLU $\Rightarrow$ LinOP) *In a market of disjoint securities, each agent having GLU with initial wealth b_i, the prices*

$$p^{(j)} = \sum_{i=1}^{N} \beta_i \Pr_i(A_j) \tag{9}$$

form a competitive equilibrium, where $\beta_i = b_i / \sum_l b_l$ is agent i's normalized initial wealth.

Proof. These prices satisfy (5) by substitution. □

This is exactly the LinOP formula (2), where the expert weights are again directly related to risk tolerance. This result can again be seen as an instance of Rubinstein's derivations [67, 68, 69].

3.1.3 Composite Agents

With disjoint securities, and under each of the utility assumptions (CARA, GLU) the market as a whole behaves as a single, *composite* agent with probabilities equal to prices. An outsider, offering securities to the group at various prices, cannot differentiate their demand response from that of an individual.

Proposition 3 (Composite CARA Agent) *Suppose that, in a market of disjoint securities, all agents exhibit CARA c_i, and effect equilibrium prices* $\mathbf{q}$. *Then their total (combined) demand for security $\langle A_j \rangle$ offered at price $p^{(j)}$ is:*

$$x_0^{(j)} = \frac{1}{c_0} \ln \left(\frac{\Pr_0(A_j)}{p^{(j)}} \cdot \frac{1 - p^{(1)} - \cdots - p^{(S)}}{\Pr_0(A_0)} \right),$$

where $\Pr_0(A_j) = q^{(j)}$ and $1/c_0 = \sum_{i=1}^{N} 1/c_i$. Thus aggregate demand $x_0^{(j)}$ is identical to that of a composite agent with beliefs $\Pr_0(A_j)$ and CARA c_0.

Proof. For each security, $\langle A_j \rangle$, the total demand of all agents is:

$$\begin{aligned}
&= \sum_{i=1}^{N} \frac{1}{c_i} \ln \left(\frac{1 - p^{(1)} - \cdots - p^{(S)}}{p^{(j)}} \cdot \frac{\Pr_i(A_j)}{\Pr_i(A_0)} \right) \\
&= \frac{1}{c_0} \ln \left(\frac{1 - p^{(1)} - \cdots - p^{(S)}}{p^{(j)}} \right) + \frac{1}{c_0} \ln \left(\frac{\prod_{i=1}^{N} [\Pr_i(A_j)]^{c_0/c_i}}{\prod_{i=1}^{N} [\Pr_i(A_0)]^{c_0/c_i}} \right) \\
&= \frac{1}{c_0} \ln \left(\frac{1 - p^{(1)} - \cdots - p^{(S)}}{p^{(j)}} \cdot \kappa \prod_{i=1}^{N} [\Pr_i(A_j)]^{\alpha_i} \cdot \frac{1}{\kappa \prod_{i=1}^{N} [\Pr_i(A_0)]^{\alpha_i}} \right) \\
&\quad \text{where } \kappa \equiv \frac{1}{\prod_{i=1}^{N} [\Pr_i(A_0)]^{\alpha_i} + \cdots + \prod_{i=1}^{N} [\Pr_i(A_S)]^{\alpha_i}} \\
&= \frac{1}{c_0} \ln \left(\frac{1 - p^{(1)} - \cdots - p^{(S)}}{p^{(j)}} \cdot \frac{q^{(j)}}{1 - q^{(1)} - \cdots - q^{(S)}} \right) \text{(from (7))} \\
&= \frac{1}{c_0} \ln \left(\frac{1 - p^{(1)} - \cdots - p^{(S)}}{p^{(j)}} \cdot \frac{\Pr_0(A_j)}{\Pr_0(A_0)} \right)
\end{aligned}$$

□

Proposition 4 (Composite GLU Agent) *Suppose that, in a market of disjoint securities, all agents exhibit GLU with initial wealth b_i, and effect*

equilibrium prices $\mathbf{q}$*. Then their total demand for security* $\langle A_j \rangle$ *offered at price* $p^{(j)}$ *is:*

$$x_0^{(j)} = b_0 \left(\frac{\Pr_0(A_j)}{p^{(j)}} - \frac{\Pr_0(A_0)}{1 - p^{(1)} - p^{(2)} - \cdots - p^{(S)}} \right),$$

where $\Pr_0(A_j) = q^{(j)}$ *and* $b_0 = \sum_{i=1}^{N} b_i$*. Thus aggregate demand* $x_0^{(j)}$ *is identical to that of a composite agent with beliefs* $\Pr_0(A_j)$ *and GLU with initial wealth* b_0*.*

Proof. Analogous to the proof of Proposition 3. □

Notice that group rationality emerges from the individual decisions of its members, unlike the classic supra Bayesian, whose existence is imposed at the outset. Furthermore, any subset of the agents can be aggregated together, interacting with the rest of the system as if it were an individual. Assuming positive c_i, the composite agent's risk aversion is strictly less than that of any individual; in other words, the group together is willing to take on more risk than any proper subset of its members. Wilson [83] first described conditions for a composite agent, which he called a *surrogate*, extended later by Rubinstein [67] and covered in Huang and Litzenberger's textbook [28]. Pratt [63] catalogs precisely when a risk-sharing group acts as a composite agent, focusing mainly on the case of homogeneous beliefs. Raiffa [64] elucidates the connection between risk sharing and group decision making, discussing the case of CARA in depth.

A mixed population, consisting of some agents with CARA and some with GLU, can be summarized as *two* composite agents, each representing its respective subpopulation. In this case, numeric solutions or approximate formulas for price are possible.

3.2 Arbitrary Securities

Deriving closed-form pricing formulae for arbitrary securities is difficult, even under homogeneous utility constraints, though a few things can be said. First, if $S = |\Omega| - 1$, and securities are linearly independent, then the market is *complete* and prices define a full probability distribution over Ω. Moreover, the distribution is irrespective of which $|\Omega| - 1$ states are chosen [40]. In particular, assuming either CARA or GLU, the consensus is the same as implied by (7) or (9), respectively, in a market with securities corresponding to all but one state.

More can be said as well when agents have CARA, and certain independence conditions among events hold.

Proposition 5 (Separation of independent events) *Suppose that an agent with CARA considers A_1 independent of all other events. Then its optimization problem for $x^{(1)}$ is separable, and reduces to the single-security case.*

Proof. Let $X = \{A_2, \ldots, A_S\}$ be the set of all events except A_1. Utility (4) is

$$\begin{aligned} & -\sum_X \left(\Pr(A_1) \Pr(X) e^{-c[(1-p^{(1)})x^{(1)} + Y_X]} + \Pr(\bar{A}_1) \Pr(X) e^{-c[-p^{(1)}x^{(1)} + Y_X]} \right) \\ = \quad & -\left(\Pr(A_1) e^{-c(1-p^{(1)})x^{(1)}} + \Pr(\bar{A}_1) e^{-c(-p^{(1)}x^{(1)})} \right) \sum_X \Pr(X) e^{-cY_X} \end{aligned}$$

where $\sum_X$ is a sum over all possible joint outcomes of events in X, and Y_X is the payoff in the selected outcome. The optimal choice for $x^{(1)}$ is independent of demand for other securities, and has the same solution as if $\langle A_1 \rangle$ were offered in isolation. □

It follows that when *all* agents have CARA, and everyone agrees that A_1 is independent of all other events, the equilibrium price is given by (7) as if $\langle A_1 \rangle$ were the only security present ($S = 1$). Moreover, because its price does not depend on the outcome of any other event, A_1 remains independent of all other events in the distribution implied by equilibrium prices. Elsewhere, we derive more general independence preservation properties of the LogOP [57], and explore the implications for securities markets [58].

In general, including the case of GLU, independence of events does *not* lead to separable demand. Even if everyone agrees that an event is independent of all others, a dependence will generally be introduced among the equilibrium prices.

Any prices that form incoherent probabilities (that allow arbitrage) cannot be part of an equilibrium. If any agent, with *any* utility function that is monotonically increasing for money, believes that some outcome is certain ($\Pr(A) \in \{0, 1\}$), or that one subset of events is logically equivalent to another, then it will effectively dictate that relationship in the consensus, by demanding infinite quantities if prices suggest otherwise.[13]

[13] Recall that in this treatment, we impose no restrictions on demand magnitude, either by budget constraints, or through market rules.

4 Learning from Prices: A Supra Bayesian Model

We have thus far assumed a classical *prior information* (PI) equilibrium in which agents' beliefs are fixed. Yet our advocacy of a market-based pooling device presupposes that exogenous prices do reveal meaningful aggregate information. If the agents themselves believe that price signals are valuable, then we would expect them to update their beliefs based on observed prices. Anecdotally, an agent whose probability is poorly aligned with the going price may begin to suspect that everyone else knows something that it does not, and adjust its belief accordingly. This is a motivating premise of the theory of *rational expectations* (RE) [24, 38]. The equilibrium conditions remain that all agents are optimizing and that total demand is zero. Implicit in the former requirement however is the dependence of each agent's *posterior* probability on the equilibrium prices. In terms of the model outlined in Section 2.3, we replace each fixed prior probability $\Pr(\cdot)$ in all equations with the conditional probability $\Pr(\cdot|\mathbf{p})$. This substitution in general transforms closed-form solutions for price into implicit equations. In some cases, explored below, we are able to resolve $\mathbf{p}$.

What is a reasonable model for $\Pr(\cdot|\mathbf{p})$? Recall that $\mathbf{p}$ has all the properties of a probability distribution and that, at least under certain homogeneity conditions, a formal equivalence between the market and a composite individual can be established. In a strong sense, the agent faces the same problem as one decision maker updating beliefs based on the probability assessments of one expert, where the "expert" in this case is the market itself. This is precisely the scenario where the supra Bayesian method (1) is both justified and well-developed. The philosophical concerns reviewed in Section 2.1 surrounding the identification of the supra Bayesian are not relevant here; moreover, some of the computational burden is reduced, since the agent updates based on only one composite distribution rather than all $N-1$ distributions. Note that this interpretation makes sense only when the agent does not consider the effect of its own decisions on prices.

Depending on how the situation is modeled, several supra Bayesian formulations are plausible [3, 6, 37, 46, 82]. We adopt one similar to that proposed by Morris [47], and later extended by Rosenblueth and Ordaz [66], as it ultimately yields closed-form results for equilibrium prices. Consider first a market with a single security $\langle A \rangle$. The agent envisions the corresponding

event A as one trial in a series of identical, independent Bernoulli trials with probability of success Π. The parameter Π is itself uncertain; assume that, after witnessing s positive outcomes or *successes* among n repetitions of the event, the agent's prior probability is $\Pr(\Pi = \pi) \propto \pi^{s-1}(1-\pi)^{n-s-1}$, or a Beta distribution with parameters s and $n-s$. The agent is myopic, or acts as if the market is a one-shot game, so uses $\Pr(A) = E[\Pi] = s/n$ to guide its decisions.

As judged by the agent, the "market" (i.e., everyone else) has observed s' successful outcomes of the event A in a total of n' trials. The likelihood function $\Pr(p|\Pi = \pi) \propto \pi^{s'}(1-\pi)^{n'-s'}$ is the corresponding Binomial distribution. For consistency, it must be that $p = s'/n'$. The desired posterior probability

$$\Pr(\Pi = \pi|p) \propto \Pr(p|\Pi = \pi)\Pr(\Pi = \pi)$$

is straightforward to compute, as the Beta and Binomial distributions form a natural conjugate class [65]. The result is another Beta distribution with parameters $s + s'$ and $n + n' - (s + s')$.

Reformulating in terms of A, we find that

$$\begin{aligned} \Pr(A|p) &= E[\Pi|p] \\ &= \frac{s+s'}{n+n'} = \frac{n}{n+n'} \cdot \frac{s}{n} + \frac{n'}{n+n'} \cdot \frac{s'}{n'} \\ &= w\Pr(A) + (1-w)p, \end{aligned} \tag{10}$$

where $w \equiv n/(n+n')$. The agent's posterior probability given the price is just a weighted average of its prior and the price. The weighting term captures the agent's perception of its own confidence, expressed in terms of number of observations, as compared to the market's.

The independent Bernoulli trials model need not be taken literally; one may prefer to interpret s/n and s'/n' as subjective measures of probability, and n and n' as measures of relative precision. In particular, the four parameters need not be constrained to the integers.

Generalize this to a market of S disjoint securities, and assume that the agents exhibit GLU. The demand function (8) becomes

$$x^{(j)}(\mathbf{p}) = wb\left(\frac{\Pr(A_j)}{p^{(j)}} - \frac{\Pr(A_0)}{1 - p^{(1)} - \cdots - p^{(S)}}\right).$$

In words, the relative stock that an agent places on its own prior probability versus public prices serves to attenuate demand by a multiplicative factor.

In a market of likewise agents, the equilibrium prices still obey (9), though these are now implicit equations:

$$\begin{aligned} p^{(j)} &= \sum_{i=1}^{N} \beta_i \Pr_i(A_j|p^{(j)}) \\ &= \sum_{i=1}^{N} \frac{w_i b_i}{\sum_l b_l} \Pr_i(A_j) + \sum_{i=1}^{N} \frac{(1-w_i) b_i}{\sum_l b_l} p^{(j)}. \end{aligned}$$

Solving for $p^{(j)}$, we find that

$$p^{(j)} = \sum_{i=1}^{N} \frac{w_i b_i}{\sum_l w_l b_l} \Pr_i(A_j). \tag{11}$$

Prices retain the form of the LinOP, but with weights proportional to the product of risk tolerance and self-assessed confidence. The introduction of a weighting factor that depends on the precision of probability assessments rather than solely on parameters of utility may be more palatable to advocates of the standard opinion pools. Why should consensus probabilities depend on utilities at all? For one, Hylland and Zeckhauser [29] prove that confounding beliefs with utilities is simply *unavoidable* if the resulting composite agent is to (1) uphold unanimous decisions, (2) maintain unanimous agreement on probabilities, and (3) allow $\Pr_0$ to depend on more than one agent. Moreover, standard opinion pools are themselves not exempt from the influence of utilities; to the contrary, all standard elicitation procedures [10, 70, 84] reveal the experts' *risk-neutral* probabilities $\Pr^{\mathrm{RN}}(\omega) \propto \Pr(\omega) u'(Y_\omega)$, rather than their true probabilities [31].

Under some conditions we can show that prices do encode the "correct" aggregate probabilities. That is, the market reaches a RE equilibrium where prices are exactly as if all agents had witnessed all observations for all events. Each agent has seen $s_i^{(j)}$ positive occurrences of A_j out of n_i total observations. Suppose that agents' observations are independent, that each agent knows its relative number of observations (i.e., $w_i = n_i / \sum_h n_h$), and that all agents have the same risk tolerance. Then (11) becomes

$$p^{(j)} = \frac{\sum_{i=1}^{N} s_i^{(j)}}{\sum_{i=1}^{N} n_i}.$$

Even though the agents' posterior probabilities are *not* in complete agreement, this is precisely the same equilibrium that would have arisen had all agents (or a composite agent) seen all observations.

Finally, analogous derivations apply when agents have CARA and the posterior update rule is a weighted *geometric* average of prior and price:

$$\Pr(A_j|p^{(j)}) \propto [\Pr(A_j)]^w [p^{(j)}]^{(1-w)}.$$

5 Wealth Dynamics and Market Adaptation

We have seen that markets can mimic both the LinOP and LogOP formulas, and that simple learning from prices can lead to confidence-based weights. These results highlight the market's flexibility, showing that popular methods for deriving consensus are in fact special cases of the market mechanism. More heterogeneous mixtures of agents entail still other pooling functions, most of which are unlikely to admit closed-form expression. Yet this begs the question of which aggregation rule is the most appropriate and/or accurate. In contrived scenarios, one or the other is clearly warranted. For example, if the agents' beliefs are distributed such that their mean is the "true" probability, then the LinOP with equal weights will recover it.[14] Similarly, if the agents' log-odds are distributed such that their mean is the true log-odds, then the LogOP is justified. Of course, it is unreasonable to assume knowledge of the distribution of agent beliefs in relation to the true probability. In this section, we argue that, over time, the market may implicitly *learn* this distribution. Markets are not only flexible but *adaptive*, rewarding accuracy and punishing inaccuracy. Agents with better probabilistic models gain wealth and, consequently, greater influence on prices.

We report simulation results in support of this claim. The market consists of a single security traded repeatedly over the course of T *periods* or rounds. With only one security per period, we can let the subscript on events A_t and the superscripts on price $p^{(t)}$ and demand $x^{(t)}$ index time. Unbeknownst to the agents, each event unfolds as an independent Bernoulli trial with probability of success π. At the beginning of time period t, the realization of A_t is unknown and agents trade in $\langle A_t \rangle$ until equilibrium. Then the outcome—either A_t or $\bar{A}_t$—is revealed, and the agents' holdings pay off accordingly. As time period $t+1$ begins, the outcome of A_{t+1} is uncertain. Agents trade in the $t+1$ period security until equilibrium, the event outcome is revealed, payoffs are collected, and the process repeats.

[14] This is precisely the construction of the comparative simulation experiments of Ng and Abramson [52]; in this light, their finding that the LinOP is best is perhaps not surprising.

Agents are *myopic*, or make decisions as if the current time period were the final one, maximizing expected utility (4) for each period in isolation. They do not update their beliefs based on observations of past events.

Agents have GLU: $u(y) = \ln(y + b)$. As discussed in Section 3.1.2, the parameter b can be construed as the agent's initial wealth in dollars. Any accumulated gain or loss is naturally reincorporated as an addition or subtraction, respectively, to this wealth parameter. Rewrite initial wealth as $b^{\langle 0 \rangle}$, and let $b^{\langle t \rangle}$ indicate the agent's wealth after period t ends. Negative wealth entails infinite disutility, so, at any given time t, the agent will never demand a quantity such that more than $b^{\langle t \rangle}$ of its dollars are at risk. All else being equal, the wealthier of two agents demands a proportionately larger quantity of the security, as dictated by (8). In an economy composed solely of agents with GLU, the equilibrium price is a wealth-weighted average (9). Thus, as an agent accrues relatively more earnings than the others, its influence on price increases. In the next two subsections, we examine how this adaptive process unfolds; first, when agents' beliefs are fixed and, second, when agents learn from price. In the former case, prices actually react as if the market were a single agent updating according to Bayes' rule.

5.1 Fixed Beliefs and the Bayesian Composite Agent

Figure 1 plots the price of the security over fifty time periods, in a market composed of one hundred agents, each with GLU, initial wealth $b_i^{(0)} = 1$, and $\Pr_i(A)$ generated randomly and uniformly on $(0, 1)$. Let $b_i^{(t)}$ indicate agent i's wealth after period t ends. For this simulation, we set $\pi = 0.5$. For comparison, the figure also shows the *observed frequency*, or the number of times that A has occurred divided by the number of periods. The market price tracks the observed frequency extremely closely. Note that price changes are due only to a transfer of wealth from inaccurate agents to accurate agents, who then wield more power in the market; individual beliefs remain fixed.

Figure 2 provides further insight into this transfer of wealth. The graph provides a snapshot of agents' wealth $b_i^{(15)}$ versus their belief $\Pr_i(A)$ after period 15. In this run, A has occurred in three out of the fifteen trials. The maximum in wealth obtains at a belief near 3/15. The solid line in the figure is a Beta distribution with parameters $3 + 1$ and $12 + 1$, scaled only by a constant. This distribution is precisely the posterior probability of success that results from the observation of 3 successes out of 15 independent Bernoulli trials, when the prior probability of success is uniform on (0,1). The

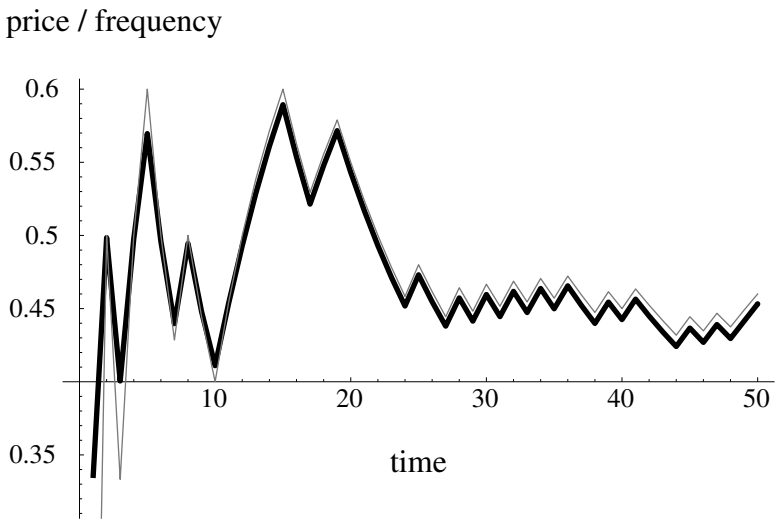

Figure 1: A single security's price (dark line) and the observed frequency (gray line) of the event over fifty time periods. The market consists of one hundred agents with GLU and $b_i^{(0)} = 1$.

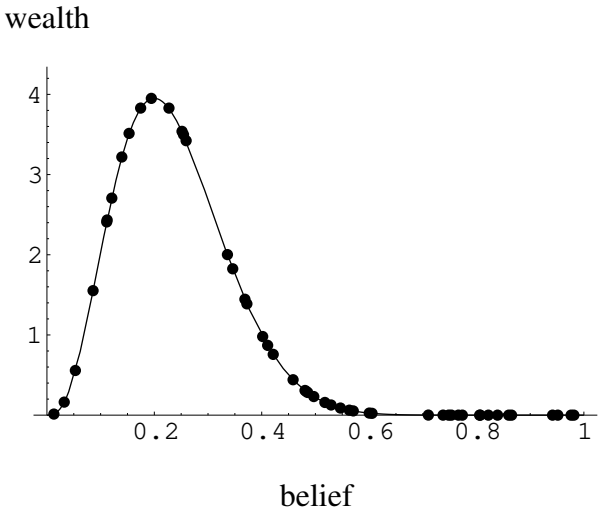

Figure 2: Wealth versus belief for fifty agents with GLU and $b_i^{(0)} = 1$, after fifteen time periods. The event has occurred in three of the fifteen trials. The solid line is the posterior Beta distribution consistent with the observation of three successes in fifteen independent Bernoulli trials.

fit is essentially perfect, and can be proved in the limit.

Proposition 6 (Market Beta) *Suppose that agents with GLU, identical initial wealth* $b_i^{(0)} = b$*, and* $\Pr_i(A)$ *generated randomly and uniformly on* $(0,1)$ *trade in a single-security market over* n *time periods. For notational brevity, let* $\pi_i \equiv \Pr_i(A)$*. Then in the limit for large* N*, the ratio of any two agents' wealth is*

$$\lim_{N\to\infty} \frac{b_h^{(n)}}{b_i^{(n)}} = \frac{(\pi_h)^s(1-\pi_h)^{n-s}}{(\pi_i)^s(1-\pi_i)^{n-s}}, \tag{12}$$

where s *is the number of positive occurrences of* A *observed in the* n *periods.*

Proof. We proceed by induction. The base case $n = 0$ holds by construction. Assume that (12) holds for $n = t$. Therefore, agent h's wealth after period t is $b_h^{(t)} = \kappa(\pi_h)^s(1-\pi_h)^{t-s}$, where κ is independent of h. The price $p^{(t+1)}$ is a wealth-weighted sum of beliefs as prescribed by (9). Because beliefs π_i are generated uniformly on (0,1), as N grows,

$$\begin{aligned}
\lim_{N\to\infty} p^{(t+1)} &= \lim_{N\to\infty} \sum_{i=1}^{N} \frac{\pi_i b_i^{(t)}}{\sum_{l=1}^{N} b_l^{(t)}} \\
&= \int_0^1 \pi \cdot \frac{\kappa(\pi)^s(1-\pi)^{t-s}}{\int_0^1 \kappa(u)^s(1-u)^{t-s}\,du}\,d\pi \\
&= E[\text{Beta}(s+1, t-s+1)] = \frac{s+1}{t+2}.
\end{aligned}$$

The agent's demand function $x^{(t+1)}\left(p^{(t+1)}\right)$ is according to (6), where $p^{(t+1)} = (s+1)/(t+2)$ as derived above. We can compute the agent's revised wealth $b_h^{(t+1)}$ after appropriate payoffs under both possible outcomes:

Case A_{t+1}**.** After a positive outcome, the agent's new wealth is $b_h^{(t+1)} = \kappa(\pi_h)^{s+1}(1-\pi_h)^{t-s}(1/p^{(t+1)})$, which is proportional to a Beta distribution reflecting one additional successful trial than at time t. The rightmost factor depends only on price, and will be identical for all agents.

Case $\bar{A}_{t+1}$**.** Revised wealth is $b_h^{(t+1)} = \kappa(\pi_h)^s(1-\pi_h)^{t-s+1}(1/(1-p^{(t+1)}))$, or proportional to a Beta distribution reflecting an additional trial without success.

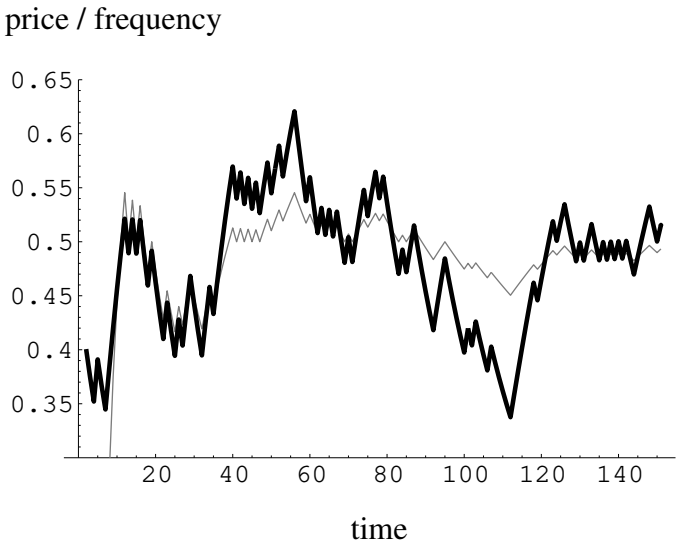

Figure 3: Price (black line) versus observed frequency (gray line) over one hundred fifty time periods for one hundred agents with GLU and learning parameter $w = 0.2$. As the frequency converges to $\pi = 0.5$, the price remains volatile.

The period $t+1$ wealth of agent i is analogous, and so (12) holds for $n = t+1$. □

Although individual agents are not adaptive, the market's composite agent appears to compute a proper Bayesian update. Specifically, wealth is reallocated proportionally to a Beta distribution corresponding to the observed number of successes and trials, and price is approximately the expected value of this Beta distribution.[15] Moreover, this correspondence holds regardless of the number of successes or failures, or the temporal order of their occurrence. A kind of collective Bayesianity *emerges* from the interactions of a group of myopic, fixed-belief agents subject to the reward structure of a securities market.

5.2 Agents that Learn

In this section, we consider agents with GLU that *learn* from the price according to the weighted average update rule (10) derived in Section 4. Figure 3 graphs the dynamics of price in an economy of of such agents, along with the observed frequency. Over time, the price remains significantly more volatile than the frequency, which converges toward $\pi = 0.5$. Below, we characterize the transfer of wealth that precipitates this added volatility; for now concentrate on the price signal itself. Inspecting Figure 3, price changes still exhibit a marked dependence on event outcomes, though at any given period the effect of recent history appears magnified, and the past discounted, as compared with the observed frequency. Working from this intuition, we attempt to fit the data to an appropriately modified measure of frequency. Define the *discounted frequency* at period n as

$$d_n = \frac{\sum_{t=1}^{n} \gamma^{n-t}(1_{A_t})}{\sum_{t=1}^{n} \gamma^{n-t}(1_{A_t}) + \sum_{t=1}^{n} \gamma^{n-t}(1_{\bar{A}_t})}, \tag{13}$$

where 1_{A_t} is the indicator function for the event at period t, and γ is the *discount factor*. Note that $\gamma = 1$ recovers the standard observed frequency.

Figure 4 illustrates a very close correlation between discounted frequency, with $\gamma = 0.96$ (hand tuned), and the same price curve of Figure 3. While standard frequency provides a provably good model of price dynamics in an economy of fixed-belief agents with GLU (Proposition 6), discounted frequency (13) appears an excellent model for GLU agents that learn from price.

Prices are dependent on the apportionment of wealth among agents and also the agents' *posterior* beliefs. Figures 5, 6, and 7 display the dependence of wealth on posterior belief $\Pr_i(A|p^{\langle t \rangle})$ at periods $t = 9$, $t = 70$, and $t = 112$, respectively, from the same experiment as in Figure 3. For comparison, each figure also shows the distribution $\texttt{Beta}(s+1, t-s+1)$ appropriate for the number of successes s observed so far in t trials. While prior beliefs are distributed uniformly on $(0, 1)$, the update rule (10) serves to compress posterior beliefs into a smaller range centered around the price. Notice that learning from prices is of most benefit to the highly inaccurate agents (in this case, near $\Pr_i(A) = \{0, 1\}$), as might be expected. Even the worst agents fare reasonably well as compared to the fixed-belief case (Figure 2), where their fortunes rapidly converge to zero. Within the reduced range of posterior

[15] As t grows, this expected value rapidly approaches the observed frequency plotted in Figure 1.

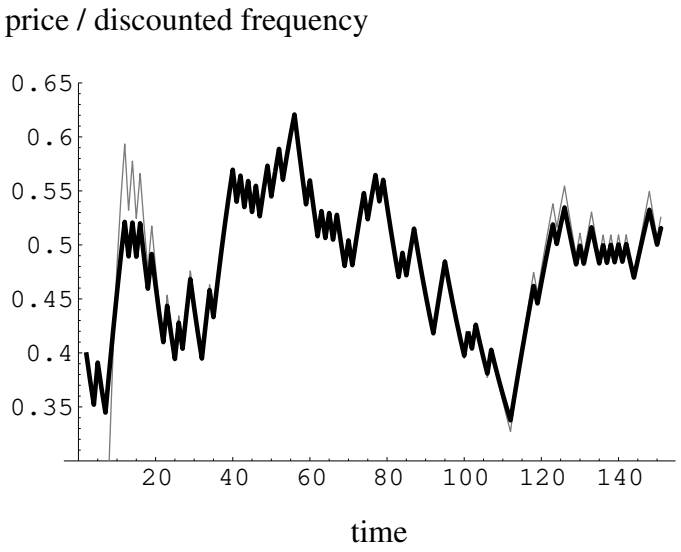

Figure 4: Price (black line) versus discounted frequency (gray line), with discount factor $\gamma = 0.96$, for the same experiment as Figure 3.

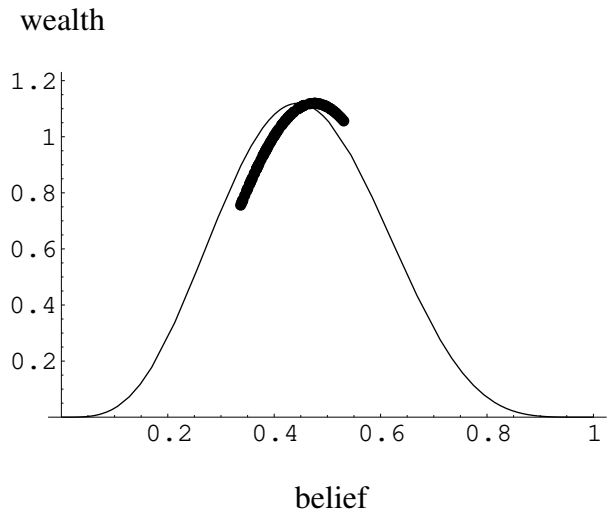

Figure 5: Wealth $b_i^{\langle 9 \rangle}$ versus belief $\Pr_i(A|p^{\langle 9 \rangle})$ at period nine of the same experiment as Figure 3. The observed frequency is $4/9$ and the solid line is $\texttt{Beta}(4+1, 5+1)$. The shapes of the two distributions match, though wealth is concentrated at the center and shifted right.

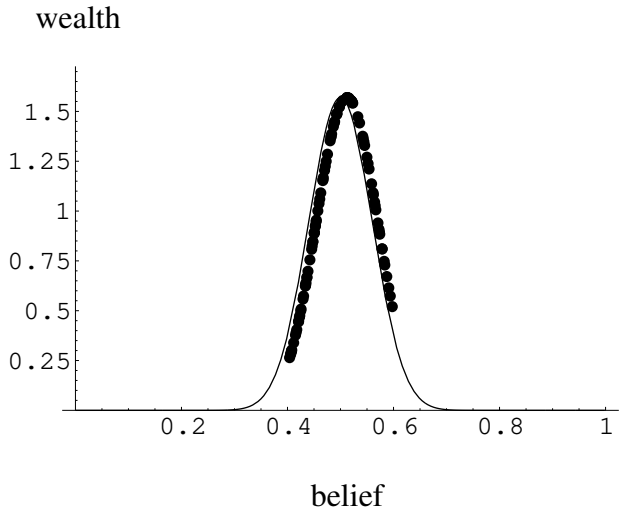

Figure 6: Wealth $b_i^{\langle 70 \rangle}$ versus belief $\Pr_i(A|p^{\langle 70 \rangle})$. The observed frequency is $35/70$ and the solid line is $\texttt{Beta}(35+1, 35+1)$. Within the window of posterior beliefs, the two distributions are well aligned.

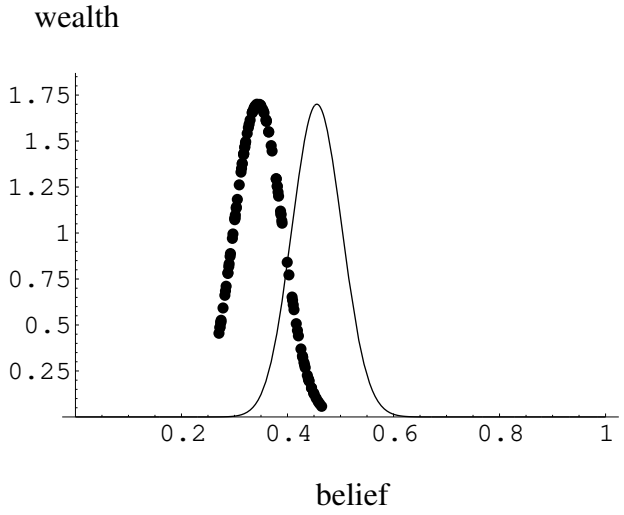

Figure 7: Wealth $b_i^{\langle 112 \rangle}$ versus belief $\Pr_i(A|p^{\langle 112 \rangle})$. The observed frequency is $51/112$ (0.455) and the solid line is $\texttt{Beta}(51{+}1, 61{+}1)$. The shapes match, but wealth is shifted significantly left, explaining the disparity between frequency and price (0.359) in Figure 3.

beliefs, the *shape* of the distribution of wealth mirrors the shape of the Beta distribution, although the position of the wealth curve may appear shifted to the left or right. The magnitude and direction of this shift correlates with features in Figure 3. For example, when the wealth distribution is shifted significantly left, as in Figure 7 (period 112), the price is considerably less than the observed frequency. In Figure 6 (period 70), the two distributions are well aligned, and price and frequency are similar.

The close fit of price to a discounted frequency can probably *not* be modeled as if the market's composite agent "forgets" long past observations, since the shape of the wealth distribution does seem to reflect all previous outcomes. Increased volatility in price arises instead from the horizontal translations evident in Figures 5, 6, and 7.

In our simulations, we observe that wealth among a subpopulation of GLU agents, mixed together also with CARA agents, continues to coincide with the appropriate Beta distribution.

6 Related Work

Eisenberg and Gale [14] propose a betting-based aggregation mechanism built not upon the securities market framework, but rather on the common method for deriving odds in horse races. They consider a pari-mutuel scheme where agents place bets across a partition of events, yielding a consensus probability equal to the proportion of the total bet on each event. If event A obtains, agents share the total amount bet in proportion to their bets on A. Agents bet to maximize expected payoffs, where expectation is with respect to their probability distribution over the events, subject to a budget constraint limiting their total bets. Eisenberg and Gale show that this mechanism yields a unique set of equilibrium probabilities, and Norvig [55] presents a dynamic process for reaching this equilibrium through iterated bids.

As the authors point out, however, this scheme can yield pathological (their word) results. For example, if there are two bettors with equal budgets, then whichever has more uniform probabilities will dictate the results. According to Genest and Zidek [23], the pari-mutuel approach to belief aggregation "has never enjoyed much popularity" for this reason. The pathological behavior, we believe, can be attributed to the role of arbitrary budgets. In the approach developed here, we impose no budgets, but rather rely on risk aversion to limit bets to the finite range.

Plott et al. [59] investigate whether pari-mutuel markets are able to aggregate information, as postulated by RE theory, in a laboratory setting. In one set of experiments, each subject was given inside knowledge that a subset of horses would definitely *not* win. Although all agents were uncertain as to the outcome, their *collective* information was enough to identify the winning horse with certainty. Information aggregation did occur, and RE-based predictions fit the data well. In a second set of experiments, subjects were given probabilistic evidence in the form of a sample of random draws from the true distribution. Some weak aggregation did occur, but not to the extent predicted by the RE model; instead, a prior information model proved more accurate.

In earlier work, Plott and Sunder [60, 61] conducted laboratory experiments to test the reasonableness of the RE assumption in the context of a securities market. Agents were initially unaware which of three states would occur. In one study [60], privileged insiders were given categorical knowledge of the underlying state. The RE model's predictions—that equilibrium prices and other economic variables converge as if everyone were aware of the true state—were significantly more accurate than those of other models, including the classical Walrasian or PI hypothesis that agents do not revise their beliefs based on prices. When insiders were given less than certain information, the results were not definitive. In a second study [61], insiders were told only that one of the three states would *not* occur. The combined knowledge of all agents was sufficient to logically infer the true state, though no single insider could directly do so. It was found that, in a complete market of three securities, the RE predictions were again the most accurate. In a single security market with agents of different preference types, the RE equilibrium was *not* realized. On the other hand, even in this last condition, Forsythe and Lundholm [15], with a similar experimental design, found that RE *was* verified, as long as subjects were experienced and preferences were common knowledge.

Beyond the controlled setting of the laboratory, researchers have investigated the aggregative properties of public markets. Hanson [27, 26] proposes an *Idea Futures* market, where participants trade in securities that pay off contingent on future developments in science, technology, or other arenas of public interest. He argues that the reward structure of such a market encourages honest revelation of opinions among scientists, and provides more accurate probability assessments for use by funding agencies, public policy leaders, the media, and other interested parties. The concept is operational

as a game (i.e., played with fake money), called the "Foresight Exchange", on the world-wide web at `http://www.ideosphere.com/`.

In a similar vein, though run with real money, the Iowa Electronic Market (IEM) (`http://www.biz.uiowa.edu/iem/`) supports trading in securities tied to the outcome of political and financial events. Their 1988 market, open only to University of Iowa students and employees, offered securities that paid off proportionally to the number of votes received by various candidates in that year's US Presidential election. The final prices predicted Bush's final margin of victory more closely than any of the six major polls [16]. Since opening to the public, subsequent US Presidential election markets have attracted wide participation and following.

West [80] presents one axiomatic justification for the (unnormalized) LogOP, based on the betting behavior of a group. He defines S_i as the value for which agent i is indifferent between D_i units of security $\langle A \rangle$, and S_i dollars for sure. Consider an assumption that the group as a whole will be indifferent between the allocation of $D = \langle D_1, \ldots, D_N \rangle$ units of the security, and the certain payoff $S = \langle S_1, \ldots, S_N \rangle$. From the premise that this is true for any D (along with some other conditions), West proves that the group's $\Pr_0(A)$ must obey an *unnormalized* geometric mean or unnormalized LogOP. In contrast to the normalized version (3), this pooling function does not in general yield a probability (e.g., if any agents disagree, $\Pr(A) + \Pr(\bar{A}) < 1$). West goes on to show that the degree by which the aggregate diverges from a probability can be used as a measure of disagreement among the agents. However, we would make the case that the group should not be indifferent between S dollars and security distribution D, unless the individual beliefs happen to be the same. In fact, the group should prefer the latter. The reason is that if the group has the security distribution, then there will be trading opportunities that can make everyone better off. We suspect that this preference can account for the normalization factor in the pooling procedure, a consequence of the market model presented here.

Varian [74, 76] analyzes the effects of divergence of opinion among agents on asset valuation and trading volume in a securities market, under various preference models. He shows that increased dispersion of beliefs is usually associated with higher volumes of trade. As long as agents' risk does not decrease too rapidly, increased dispersion also coincides with lower prices for securities.

7 Conclusion

We have outlined a methodology for pooling opinions based on individually rational agents (utility maximizers) trading in a competitive securities market, where the resultant equilibrium prices represent consensus beliefs. Our framework is motivated from the bottom up: we begin with several common assumptions about agents (mainly that each has a subjective probability distribution, has risk averse utility for money, and behaves competitively), and seek to analyze properties of the implied price equilibrium. We establish a few properties for general markets, and provide closed-form characterizations of demand and equilibrium prices for the two cases of negative exponential and generalized logarithmic utility for money. The expert weights in the standard pools coincide with normalized measures of risk tolerance in our model, providing a decision-theoretic interpretation for an often ungrounded concept.

We further explored the possibility that agents update their beliefs based on price feedback. In a simple model of such behavior, where agents can express their own as well as the market's information in terms of equivalent samples, their posterior probabilities are a weighted average of their prior beliefs and the observed market price. Given our standard utility models, the resulting equilibrium prices retain the same functional form, but the expert weights depend on the agents' own assessments of their relative qualities of information.

We investigated a complementary situation, where the agents do not learn, but the market as a whole adapts over time. Price fluctuations are due solely to redistribution of wealth from historically inaccurate agents to those enjoying relative success. In a one-security model, under particular assumptions, these price changes can be rationalized as the Bayesian updates of a representative agent witnessing the outcomes of repeated Bernoulli trials, thus extending Rubinstein's [67] notion of a composite agent to the multiperiod case. When agents learn from prices in a multiperiod market, price fluctuations adhere more closely to a model of discounted frequency than to the pure observed frequency.

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