Performance of prediction markets with Kelly bettors

David Pennock (Yahoo!) with John Langford (Yahoo!) & Alina Beygelzimer (IBM)
A prediction market

2011 the warmest year on record? (Y/N)

I am entitled to:

<table>
<thead>
<tr>
<th></th>
<th>2011 the warmest</th>
<th>not the warmest</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 if</td>
<td>$0 if</td>
<td></td>
</tr>
</tbody>
</table>

http://intrade.com

<table>
<thead>
<tr>
<th>Contract</th>
<th>Bid</th>
<th>Ask</th>
<th>Last</th>
<th>Vol</th>
<th>Chge</th>
</tr>
</thead>
<tbody>
<tr>
<td>2011.GLOBALTEMP.WARMEST</td>
<td>4.5</td>
<td>6.7</td>
<td>5.0</td>
<td>2599</td>
<td>0</td>
</tr>
</tbody>
</table>

2011 Sep 15 - 9:16PM GMT
Wisdom of crowds

Expected score vs Number of experts aggregated (2004 data)

When to guess: if you’re in the 99.7th percentile

More:
http://www.overcomingbias.com/2007/02/how_and_when_to.html
Can a prediction market learn?

Typical
- Mkt >> avg expert
- Static

This paper
- Mkt >? best expert
- Dynamic/Learning (experts algos)
- Assumption: Agents optimize growth rate (Kelly)
• You have $1000
• Market price mp is 0.05
• You think probability p is 0.10
• Q: How much should you bet?
• Wrong A: $1000
  P(bankruptcy)→1
Kelly betting

- You have $1000
- Market price $mp$ is 0.05
- You think probability $p$ is 0.10
- Q: How much should you bet?
- Wrong A: fixed $1$
  No compounding magic
• You have $1000
• Market price mp is 0.05
• You think probability p is 0.10
• Q: How much should you bet?
• Optimal A: $52.63
  or f* fraction of your wealth
  where f* = (p-mp)/(1-mp)  “edge/odds”
Why?

• Kelly betting \equiv \text{maximizing log utility}
  
  – Maximizes compounding growth rate
    Maximizes geometric mean of wealth
  
  – Minimizes expected doubling time
    Minimizes exp time to reach, say, $1M
  
  – Does not maximize expected wealth
    (“All in” does, but ensures bankruptcy)
Fractional Kelly betting

- Bet $\lambda f^*$ fraction of your wealth, $\lambda \in [0,1]$
- Why? Ad hoc reasons
  - Full Kelly is too risky (finite horizon)
  - I’m not confident of $p$ (2nd-order belief)
- Our (new) reason
  - $\lambda$ fraction Kelly behaviorally equiv to full Kelly with revised belief $\lambda p + (1-\lambda)mp$
  - Has Bayesian justification
• Assume all agents optimize via Kelly and are “price takers”

• Then $mp = \sum w_i p_i$

• Fractional Kelly: $mp = \sum \lambda_i w_i p_i$
Wealth dynamics

• Agents trade in a binary pred market
  Event outcome is revealed
  Wealth is redistrib. (losers pay winners)

Repeat

• After each round
  If event happens: \( w_i \leftarrow w_i \frac{p_i}{pm} \)
  If event doesn’t: \( w_i \leftarrow w_i \frac{(1-p_i)}{(1-pm)} \)

• Wealth is redistributed like Bayes rule!
Wealth dynamics

Fig. 1. a) Price (black line) versus the observed frequency (gray line) of the event over fifty time periods. The market consists of one hundred full-Kelly agents with initial wealth $w_i = 1$. b) Wealth after fifteen time periods versus belief for fifty Kelly agents. The event has occurred in three of the fifteen trials. The solid line is the posterior Beta distribution consistent with the observation of three successes in fifteen independent Bernoulli trials.
Wealth dynamics: Regret

• Theorem:
  \[ \text{Mkt log loss} < \min_i \text{agent } i \text{ log loss} - \ln w_i \]

• Applies for all agents, all outcomes, even adversarial

• Same bound as experts algorithms
  \[ \Rightarrow \text{No “price of anarchy”} \]
Learning the Kelly fraction

• Competitive equilibrium: $\lambda = 1$
• Rational expectations equil: $\lambda = 0$
• Practical (wisdom of crowds): $\lambda = \varepsilon$
• A proposal: Fractional Kelly as an experts algorithm btw yourself and the market
  – Start with $\lambda = 0.5$
  – If right, increase $\lambda$; If wrong, decrease
  – Won’t do much worse than market (0)
    Won’t do much worse than original prior $p$
Self-interested Kelly bettors implement Bayes’ rule, minimize market’s log loss

If mkt & agents care about log loss: win-win

Questions

– Can non-Kelly bettors be induced to minimize log loss?
– Can Kelly bettors be induced to minimize squared loss, 0/1 loss, etc.?
– Can/should we encourage Kelly betting?